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G reductive gp

root datum $(\Lambda, \Lambda^*, \Phi, \check{\Phi})$ where $\Lambda = \mathrm{Hom}(\mathbb{G}_m, T)$ and $\Lambda^* = \mathrm{Hom}(T, \mathbb{G}_m)$

$\Phi \subset \Lambda^*$
$\check{\Phi} \subset \Lambda$

<u>Lemma.</u> If $\alpha \in \Phi$ has the form $\alpha = 2\beta$ with $\beta \in \Lambda^*$, then α is a long root of a direct factor $Sp_{2n} \subset G = Sp_{2n} \times G'$. $(n \geq 1)$

Proof. $\alpha \in \underset{\uparrow}{\Phi^{irr}} \subset \Phi$ ← irred. root system. wlog α simple (choosing a suitable Borel)

- Φ^{irr} cannot be of type A_n $(n \geq 2)$, D_n $(n \geq 3)$, E_6, E_7, E_8 or G_2 since then $\exists\, \gamma \in \check{\Phi}$ with $\langle \alpha, \gamma \rangle$ odd
- α can't be a short root in B_n, C_n or F_4
- — " — long root in B_n $(n \geq 3)$ or F_4

$\Longrightarrow$ α has to be a long root in C_n

Let $\Delta = \{\text{simple roots in } \Phi^{irr}\} \ni \alpha$

We know
$$\det\left(\langle \gamma, \check{\delta} \rangle\right)_{\substack{\gamma \in \Delta \\ \check{\delta} \in \check{\Delta}}} = 2 \quad \text{for } C_n$$

Replace α by $\beta = \frac{1}{2}\alpha$, then $\det(\cdots) = 1$.

Can split

$$\Lambda^* = \overbrace{\left(\mathbb{Z}\cdot\beta + \sum_{\substack{\gamma\in\Delta \\ \gamma\neq\alpha}} \mathbb{Z}\cdot\gamma\right)}^{\Lambda^*(Sp_{2n}) \supset \Phi^{irr}} \oplus \check{\Delta}^{\perp}$$

$$\Lambda = \quad \dots \quad \oplus \quad \dots$$

$$\Longrightarrow \Phi = \Phi^{irr} \,\dot\cup\, \Phi^{rest}$$

$\square$

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$$\begin{array}{c} \tilde{C} \\ \pi \downarrow \\ C \end{array} \quad \text{cameral cover}$$

G red. gp $/ k = \bar{k}$ $\qquad T \subset G$ max. torus

char$(k) \nmid |W|$ $\qquad$ root datum $X_* = \mathrm{Hom}(\mathbb{G}_m, T) \supset \check{\Phi}$

$$X^* = \mathrm{Hom}(T, \mathbb{G}_m) = \mathrm{Hom}(X_*, \mathbb{Z})$$

Sheaves of groups on C:

- $J \subset J^1 = (\pi_* T)^W$

  $c \in C$ unramified $\Rightarrow J^1_c = (\mathbb{Z}[W] \otimes_{\mathbb{Z}} T)^W \cong T$

- $J^0 :=$ (fibrewise) connected component of J^1

  $c \in C$ unramified $\Rightarrow J^0_c = J_c = J^1_c$

• For $\alpha \in \Phi$, put $\tilde{C}^\alpha := \tilde{C}^{s_\alpha} \overset{\pi}{\twoheadrightarrow} C^\alpha$. Then:

For $c \in C^\alpha$ we have
$$J_c \cong \ker(\alpha : T \to \mathbb{G}_m)$$
$$J_c^1 \cong \ker(\check{\alpha}\circ\alpha : T \to T)$$
$$J_c^\circ \cong \ker(\alpha : T \to \mathbb{G}_m)^\circ .$$

Recall:
$$\mu_{\check{\alpha}} := \ker \check{\alpha} = \begin{cases} \{1\} \\ \{\pm 1\} \end{cases}$$

$$m_{\check{\alpha}} := |\mu_{\check{\alpha}}|$$

$$\check{\alpha}_{red} := \frac{1}{m_{\check{\alpha}}} \cdot \check{\alpha}$$

$$\mu_\alpha := (\mathbb{Q}\alpha \cap X^*)/\mathbb{Z}\alpha$$

$$m_\alpha := |\mu_\alpha|$$

$$\alpha_{red} := \frac{1}{m_\alpha} \cdot \alpha \qquad \Longrightarrow \quad J_c^\circ = \ker(\alpha_{red}) \text{ for } c \in C^\alpha.$$

Last lemma. If $m_\alpha \neq 1$, then $m_\alpha = 2$ and α is a long root for a direct factor Sp_{2n}, $n \geq 1$.

If $m_{\check{\alpha}} \neq 1$, then $m_{\check{\alpha}} = 2$ and α is a short root for a direct factor SO_{2n+1}, $n \geq 1$.

Recall:
$$1 \to J \to J^1 \to \underbrace{\left(\prod_\alpha \pi_*(\mu_{\check{\alpha}}|_{\tilde{C}^\alpha})\right)^W}_{=: \square_1} \to 1$$

Analogously

$$1 \longrightarrow \mathcal{J}^0 \longrightarrow \mathcal{J} \longrightarrow \underbrace{\Big(\prod_\alpha \pi_* (\mu_\alpha | \tilde{C}_\alpha) \Big)^W}_{=: \, \square_0} \longrightarrow 1$$

Let $\mathcal{P}, \mathcal{P}^1, \mathcal{P}^0$ be the stacks of $\mathcal{J}, \mathcal{J}^1, \mathcal{J}^0$ - torsors.

Get an exact sequence of Picard stacks

$$0 \longrightarrow \square_1 \longrightarrow \mathcal{P} \longrightarrow \mathcal{P}^1 \longrightarrow 0$$

$$0 \longrightarrow \square_0 \longrightarrow \mathcal{P}^0 \longrightarrow \mathcal{P} \longrightarrow 0$$

<u>Claim:</u> $(\mathcal{P})^\vee \simeq \check{\mathcal{P}}$

<u>Philosophy:</u> Prove $(\mathcal{P}^0)^\vee \simeq \check{\mathcal{P}}^1$
$(\mathcal{P}^1)^\vee \simeq \check{\mathcal{P}}^0$ and reduce to these cases.

Coarse moduli spaces $\mathcal{P}_{coarse} \cong H^1(C, \mathcal{J})$, $\mathcal{P}^0_{coarse}$, $\mathcal{P}^1_{coarse}$

with connected comp^{ts} $P \subset \mathcal{P}_{coarse}$, $P^0 \subset \mathcal{P}^0_{coarse}$, $P^1 \subset \mathcal{P}^1_{coarse}$

Have already reduced $(\check{\mathcal{P}})^\vee \simeq \mathcal{P}$ to $(\check{P})^\vee \simeq P$.

Strategy: Construct iso $(\check{P}^1)^\vee \simeq P^0$

so that we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (\check{A}')^\vee \arrow[r] \arrow[d, "\wr\lambda"] & (\check{P}^1)^\vee \arrow[r] \arrow[d, "\wr"] & (\check{P})^\vee \arrow[r] \arrow[d, "\exists"] & 1 \\
0 \arrow[r] & A \arrow[r] & P^0 \arrow[r] & P \arrow[r] & 1
\end{tikzcd}$$

where A, $(\check{A}')^\vee$ are the kernels.

It is sufficient to prove that ι_A is surjective (being injective anyway), or equivalently:

$$|\check{A}'| = |(\check{A}')^\vee| \overset{!}{\geqq} |A| \quad \text{where } \check{A}' := \ker(\check{P} \to \check{P}^1).$$

Recall

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & 0 \arrow[d] \\
 & & 0 \arrow[r] & A' \arrow[r] & P \arrow[r] \arrow[d] & P^1 \arrow[d] \\
1 \arrow[r] & Z \arrow[r] & Z^1 \arrow[r] \arrow[d, equal] & \square_1 \arrow[r] \arrow[dr, dashed] & \wp_{coarse} \arrow[r] \arrow[d] & \wp^1_{coarse} \arrow[r] \arrow[d] & 0 \\
 & & T^W & & \pi_0(\wp) \arrow[r] \arrow[d] & \pi_0(\wp^1) \arrow[d] \\
 & & & & 0 & 0
\end{tikzcd}$$

Define $A'_\infty := \ker(\square_1 \longrightarrow \pi_0(\wp))$.

We get exact sequences

$$1 \longrightarrow A'_\infty \longrightarrow \square_1 \longrightarrow \pi_0(\wp) \longrightarrow \pi_0(\wp^1) \longrightarrow 1$$

$$\begin{tikzcd}
1 \arrow[r] & Z \arrow[r] & Z^1 \arrow[r] \arrow[d, equal] & A'_\infty \arrow[r] & A' \arrow[r] & 0 \\
 & & T^W & & &
\end{tikzcd}$$

Conclusion: $$|A'| = \frac{|\square_1|}{|\mathrm{coker}(Z \to Z^1)| \cdot |\ker(\pi_0(\wp) \to \pi_0(\wp^1))|}$$

Analogously: $$|A| = \frac{|\square_0|}{|\mathrm{coker}(Z^0 \to Z)| \cdot |\ker(\pi_0(\wp^0) \longrightarrow \pi_0(\wp))|}$$

To prove $|\check{A}'| \geq |A|$ we claim:

(1) $|\check{\square}_1| = |\square_0|$,

(2) $|\operatorname{coker}(\check{Z} \to \check{Z}^1)| = |\ker(\pi_0(\mathcal{P}^0) \longrightarrow \pi_0(\mathcal{P}))|$,

(3) $|\ker(\pi_0(\check{\mathcal{P}}) \to \pi_0(\check{\mathcal{P}}^1))| \leqq |\operatorname{coker}(Z^0 \to Z)|$.

Put $Z^0 := \bigcap\limits_{\alpha \in \Phi} \ker(\alpha_{red} : T \to \mathbb{G}_m)$

$= \{ x \in Z \mid x = 1$ in each direct factor of type $Sp_{2n} \}$.

Proof. (1) clear by construction.

(2) Recall $\pi_0(\mathcal{P}) = \pi_1(G) = X_* / \mathbb{Z}\check{\Phi}$,

analogously $\pi_0(\mathcal{P}^0) = X_* / \sum\limits_{\alpha \in \Phi} \check{\alpha}(\alpha(X_*))$

$$\Rightarrow \quad |\ker \pi_0(\mathcal{P}^0) \to \pi_0(\mathcal{P})| = 2^{\#(\{\alpha \in \Phi \mid m_\alpha = 2\}/W)}$$

$$= |\operatorname{coker}(\check{Z} \longrightarrow \underset{\check{T}^W}{\underset{\|}{\check{Z}^1}})|.$$

(3) <u>Lemma.</u> If $G = G_a \times G_b$

then $\mathcal{J}^1 = \mathcal{J}^1_a \times \mathcal{J}^1_b$, $\mathcal{J} = \mathcal{J}_a \times \mathcal{J}_b$, ...

$\pi_0(\mathcal{P}^1_G) = \pi_0(\mathcal{P}^1_{G_a}) \times \pi_0(\mathcal{P}^1_{G_b})$, ...

Proof of the lemma.

Have Cartesian square of Cameral covers

$$\begin{tikzcd} & \tilde{C}_G \arrow[dl, "\tilde{\pi}_a"'] \arrow[dr, "\tilde{\pi}_b"] & \\ \tilde{C}_{G_a} =: \tilde{C}_a \arrow[dr, "\pi_a"'] \arrow[rr, phantom, "\diamond"] & & \tilde{C}_b \arrow[dl, "\pi_b"] \\ & C & \end{tikzcd}$$

$T = T_a \times T_b$

$$\mathcal{J}^1 = (\pi_* T)^W = (\pi_* T_a \times T_b)^{W_a \times W_b} = (\pi_{a*} \tilde{\pi}_{b*} T_a) \times (\pi_{b*} \tilde{\pi}_{a*} T_b)^{W_a \times W_b}$$

$$= \left(\pi_{a*} (\tilde{\pi}_{b*} T_a)^{W_b}\right)^{W_a} \times \dots$$

$$= (\pi_{a*} T_a)^{W_a} \times (\pi_{b*} T_b)^{W_b}$$

$$= \mathcal{J}^1_a \times \mathcal{J}^1_b. \qquad \square$$

May now reduce the claim (3) to the case where G is indecomposable.

If $G \not\cong Sp_{2n} \Rightarrow$ both sides are $= 1$ ✓

If $G \cong Sp_{2n} \Rightarrow |\ker(\pi_0(\check{\mathcal{P}}) \to \pi_0(\check{\mathcal{P}}^1))| \leq |\pi_0(\check{\mathcal{P}})|$

$$|\pi_0(\check{\mathcal{P}})| = |\pi_1(SO_{2n+1})| = 2$$

while

$|\operatorname{coker}(Z^0 \to Z^1)| = 2.$ ✓ $\square$

<u>Remarks.</u> • $\pi_0(\mathcal{P}^1) = X_* / \sum_{\alpha \in \Phi} (\mathbb{Q}\check{\alpha} \cap X_*)$ follows if duality is proved.

• For $\mathcal{P}^0$ one has exponential sequence $/\mathbb{C}$ resp. Kummer sequence $/$char p

$$1 \to \mathcal{J}^0[n] \to \mathcal{J}^0 \xrightarrow{n} \mathcal{J}^0 \to 1, \quad \left(\begin{matrix} p \nmid n, \text{ e.g.} \\ n = \ell^r \end{matrix}\right)$$

- Have $P^1 = \left(\left(\mathrm{Jac}(\tilde{C}) \otimes_{\mathbb{Z}} X_* \right)^W \right)^0 \hookrightarrow \mathrm{Jac}(\tilde{C}) \otimes_{\mathbb{Z}} X_*$

- For $U := C \setminus \bigcup_{\alpha \in \Phi} C^\alpha \xhookrightarrow{j} C$,

consider
$$\begin{tikzcd} \tilde{U} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \tilde{C} \arrow[d, "\pi"] \\ U \arrow[r, hook, "j"'] & C \end{tikzcd}$$
and $(\pi_* X_*|_{\tilde{U}})^W =: L$

then L is a local system on U in the analytic topology.

$\mathcal{J}, \mathcal{J}^0, \mathcal{J}^1$ sheaf of analytic sections of J, J^0, J^1

Claim: $\mathcal{J}^0 = (j_* L) \otimes \mathcal{O}_C^*$

$\mathcal{J}^1 = j_*(L \otimes \mathcal{O}_U^*)$.